So far in our discussion of rotations in 3D we have encountered scalars, vectors and tensors. These are 0, 3 and higher dimensional representations of the rotation group.

What about a 2D representation of 3D rotations?

We would need 2x2 matrices satisfying $[g_i, g_j] = i\epsilon^{ijk} g_k$ where $\underbrace{i,j,k = 1,2,3}_{3D}$ $[g_1,g_2] = ig_3$, $[g_2,g_3] = ig_1$, $[g_3,g_1] = ig_2$

These work: $g_{R_{yz}} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ $g_{R_{zx}} = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$ $g_{R_{xy}} = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

$= \frac{1}{2}\sigma_x$, $= \frac{1}{2}\sigma_y$, $= \frac{1}{2}\sigma_z$ where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices

Now we can build: $R_{yz}(\theta) = e^{ig_{R_{yz}}\theta} = \underbrace{\begin{pmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}}$ and similarly for R_{zx} and R_{xy}.

Often we write $\chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$ satisfy $U^+U = \mathbb{I}$ and $\det U = +1$ $\Big\}$ $SU(2)$ which act on complex 2-component spinors χ.

So $SO(3) \sim SU(2)$, at least near the identity (which is all the Lie algebra knows about).

Globally however there is a difference: $SO(3)$ $R_x(2\pi) = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} = \mathbb{I}$
$SU(2)$ $R_x(2\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{I}$ $\Big\}$ $SU(2)$ is called the double-cover of $SO(3)$

of course $R_x(4\pi) = \mathbb{I}$ for both!

There is a certain sense in which spinors and $SU(2)$ probes geometry more deeply than coordinates, scalars, vectors, $SO(3)$, etc.
By "probe more deeply" I mean they contain more information. Sometimes people say that spinors know about the square root of the geometry.
In fact if we consider the anti-commutator of the Pauli matrices we find: $\underbrace{\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_{2\times2}}_{\text{A Clifford algebra}}$

Example: $\sigma_x\sigma_y + \sigma_y\sigma_x = \begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix}\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix} + \begin{pmatrix} 0&-i \\ i&0 \end{pmatrix}\begin{pmatrix} 0&1 \\ 1&0 \end{pmatrix} = \begin{pmatrix} i&0 \\ 0&-i \end{pmatrix} + \begin{pmatrix} -i&0 \\ 0&i \end{pmatrix} = \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$ as expected since $\delta_{xy} = 0$
$\sigma_y\sigma_y + \sigma_y\sigma_y = 2\sigma_y\sigma_y = 2\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix}\begin{pmatrix} 0&-i \\ i&0 \end{pmatrix} = 2\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$ as expected since $\delta_{yy} = 1$

It might seem silly, but recall that $\delta_{ij} = \begin{pmatrix} 1 && \\ & 1 & \\ && 1 \end{pmatrix}$ is the metric of $\mathbb{R}^3$. This will come in handy later.

Another illustration of this is a lesson from QM: If we only have integer spin states at our disposal, $\{0,1,2,\cdots\}$ then by combining spins we can only ever build more integer spin states.
However if we allow ½ integer spin states, then we can build ½ or whole integer states just using ½ spin states, e.g. $\frac{1}{2} - \frac{1}{2} = 0$, $\frac{1}{2} + \frac{1}{2} = 1$.

To finish up, we need to determine how to build an invariant (for Lagrangians) out of spinors.

Following our usual recipe: If $\chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$ then we define $\tilde{\chi} \to \tilde{\chi}' = \left(e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\right)^{-1\,T}\tilde{\chi}$

then $\tilde{\chi}^T\chi$ is invariant.

$*T$

But recall how we form $\tilde{\chi}$ from χ: $\tilde{\chi} = (g\chi)^*$ where $\left(e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\right)^{\dagger} g\, e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} = g$

However for SU(2) we already know that $U^\dagger U = 1$ so $g = I$ and

we can say $\tilde{\chi} = (g\chi)^* = \chi^*$ and then $\chi^{*T}\chi = \chi^\dagger\chi$ is invariant!

Note: All of the σ matrices are Hermitian, i.e. $\sigma_i^\dagger = \sigma_i$, $\vec{\theta}$ is real so

$$U^\dagger = \underbrace{\left(e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\right)^\dagger = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}}_{\text{You can see more explicitly by Taylor expanding}} = U^{-1}$$

This will not be the case later!

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Now it is time to repeat this procedure for special relativity.

The Lorentz transformations as they act on coordinates/vectors form $SO(1,3)$ so let's explore its algebra.

We expect 6 generators corresponding to: $\underbrace{R_{yz}, R_{zx}, R_{xy}}, \underbrace{B_{xt}, B_{yt}, B_{zt}}$.

We will call the corresponding generators: $J_1, J_2, J_3 \quad K_1, K_2, K_3$

Fortunately we already know a lot about the J's :

$$J_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix} \Rightarrow [J_i, J_j] = i\epsilon^{ijk} J_k$$

From which we can also get $SU(2)$

If we take the various boosts and again consider their Taylor expansion, then using the exponential map $B = \exp(iK\delta B)$ we find:

$$K_1 = \begin{pmatrix} 0&-i&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0&0&0&-i \\ 0&0&0&0 \\ 0&0&0&0 \\ -i&0&0&0 \end{pmatrix}$$

Now is where it gets interesting. By brute force one can show:

$[K_i, K_j] = -i\epsilon^{ijk} J_k$ 2 boosts $\rightarrow$ rotation

$[J_i, K_j] = i\epsilon^{ijk} K_k$ rotation + boost = boost

Question: Can the boosts alone form a subgroup of $SO(1,3)$? No
What about rotations? Yup

So unfortunately the boosts and rotations of $SO(1,3)$ do not cleanly split from each other.

But...